

INFINITE-SAMPLE CONSISTENT ESTIMATIONS OF PARAMETERS OF THE WIENER PROCESS WITH DRIFT

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ABSTRACT. We consider the Wiener process with drift

$$dX_t = \mu dt + \sigma dW_t$$

with initial value problem $X_0 = x_0$, where $x_0 \in R$, $\mu \in R$ and $\sigma > 0$ are parameters. By use values $(z_k)_{k \in N}$ of corresponding trajectories at a fixed positive moment t , the infinite-sample consistent estimates of each unknown parameter of the Wiener process with drift are constructed under an assumption that all another parameters are known. Further, we propose a certain approach for estimation of unknown parameters x_0, μ, σ of the Wiener process with drift by use the values $(z_k^{(1)})_{k \in N}$ and $(z_k^{(2)})_{k \in N}$ being the results of observations on the $2k$ -th and $2k + 1$ -th trajectories of the Wiener process with drift at moments t_1 and t_2 , respectively.

1. INTRODUCTION

Following [1], the Wiener process with drift is used as a mathematical model described a random motion of a particle suspended in water which is being bombarded by water molecules. The temperature of the water will influence the force of the bombardment, and thus we need a parameter σ to characterize this. Moreover, there is a water current which drives the particle in a certain direction, and we will assume a parameter μ to characterize the drift. To describe the displacements of the particle, the Wiener process can be generalized to the process

$$dX_t = \mu dt + \sigma dW_t \tag{1.1}$$

which has solution

$$X_t = x_0 + \mu t + \sigma W_t \tag{1.2}$$

for $X_0 = x_0$ (see [1], p.11). It is thus normally distributed with mean $x_0 + \mu t$ and variance $\sigma^2 t$, as follows from the properties of the standard Wiener process. This process has been proposed as a simplified model for the membrane potential evolution in a neuron.

The parameters in (1.1)-(1.2) have the following sense:

- (i) μ represents the equilibrium or mean value supported by fundamentals (in other words, the central location) and called a parameter of the drift in Wiener model with drift ;
- (ii) σ is a parameter of the bombardment force in Wiener model with drift ;
- (iii) x_0 is an initial position of the particle suspended in water in Wiener model with drift ;

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(iv) x_t is a position of the particle suspended in water in Wiener model with drift at moment $t > 0$;

The purpose of the present paper is to introduce a new approach which by use values $(z_k)_{k \in N}$ of corresponding trajectories at a fixed positive moment t , will allows us to construct a consistent estimate for each unknown parameter of the Wiener model with drift under an assumption that all another parameters are known. Note that analogous problem has been considered by L.Labadze and G. Pantsulaia for Ornstein-Uhlenbeck's stochastic process (cf. [2]).

The rest of the present paper is the following:

In Section 2 we consider some auxiliary notions and facts from the theory of mathematical statistics and probability.

In Section 3 we present the constructions of consistent and infinite-sample consistent estimates for unknown parameters of the Wiener model with drift.

In Section 4 we present simulations and animations of the Wiener model with drift.

In Section 5 we propose a certain approach for estimation of unknown parameters x_0, μ, σ of the Wiener process with drift by use the values $(z_k^{(1)})_{k \in N}$ and $(z_k^{(2)})_{k \in N}$ being the results of observations on the $2k$ -th and $2k + 1$ -th trajectories of the Wiener process with drift at moments t_1 and t_2 , respectively.

2. SOME AUXILIARY NOTIONS AND FACTS

We begin this subsection by the following definition.

Let $\{\mu_\theta : \theta \in R\}$ be a family Borel probability measures in R . By μ_θ^N we denote the N -power of the measure μ_θ for $\theta \in R$.

Definition 2.1. A Borel measurable function $T_n : R^n \rightarrow R$ ($n \in N$) is called a consistent estimator of a parameter θ (in the sense of everywhere convergence) for the family $(\mu_\theta^N)_{\theta \in R}$ if the following condition

$$\mu_\theta^N(\{(x_k)_{k \in N} : (x_k)_{k \in N} \in R^N \text{ \& } \lim_{n \rightarrow \infty} T_n(x_1, \dots, x_n) = \theta\}) = 1$$

holds true for each $\theta \in R$.

Definition 2.2. A Borel measurable function $T_n : R^n \rightarrow R$ ($n \in N$) is called a consistent estimator of a parameter θ (in the sense of convergence in probability) for the family $(\mu_\theta^N)_{\theta \in R}$ if for every $\epsilon > 0$ and $\theta \in R$ the following condition

$$\lim_{n \rightarrow \infty} \mu_\theta^N(\{(x_k)_{k \in N} : (x_k)_{k \in N} \in R^N \text{ \& } |T_n(x_1, \dots, x_n) - \theta| > \epsilon\}) = 0$$

holds.

Definition 2.3. A Borel measurable function $T_n : R^n \rightarrow R$ ($n \in N$) is called a consistent estimator of a parameter θ (in the sense of convergence in distribution) for the family $(\mu_\theta^N)_{\theta \in R}$ if for every continuous bounded real valued function f on R the following condition

$$\lim_{n \rightarrow \infty} \int_{R^N} f(T_n(x_1, \dots, x_n)) d\mu_\theta^N((x_k)_{k \in N}) = f(\theta)$$

holds.

Remark 2.1 Following [3] (see, Theorem 2, p. 272), for the family $(\mu_\theta^N)_{\theta \in R}$ we have:

(a) an existence of a consistent estimator of a parameter θ in the sense of everywhere convergence implies an existence of a consistent estimator of a parameter θ in the sense of convergence in probability;

(b) an existence of a consistent estimator of a parameter θ in the sense of convergence in probability implies an existence of a consistent estimator of a parameter θ in the sense of convergence in distribution.

Definition 2.4 Following [4], the family $(\mu_\theta^N)_{\theta \in R}$ is called strictly separated if there exists a family $(Z_\theta)_{\theta \in R}$ of Borel subsets of R^N such that

- (i) $\mu_\theta^N(Z_\theta) = 1$ for $\theta \in R$;
- (ii) $Z_{\theta_1} \cap Z_{\theta_2} = \emptyset$ for all different parameters θ_1 and θ_2 from R .
- (iii) $\cup_{\theta \in R} Z_\theta = R^N$.

Definition 2.5. Following [4], a Borel measurable function $T : R^N \rightarrow R$ is called an infinite sample consistent estimator of a parameter θ for the family $(\mu_\theta^N)_{\theta \in R}$ if the following condition

$$(\forall \theta)(\theta \in R \rightarrow \mu_\theta^N(\{(x_k)_{k \in N} : (x_k)_{k \in N} \in R^N \text{ \& } T((x_k)_{k \in N}) = \theta\}) = 1)$$

holds.

Remark 2.2. Note that an existence of an infinite sample consistent estimator of a parameter θ for the family $(\mu_\theta^N)_{\theta \in R}$ implies that the family $(\mu_\theta^N)_{\theta \in R}$ is strictly separated. Indeed, if we set $Z_\theta = \{(x_k)_{k \in N} : (x_k)_{k \in N} \in R^N \text{ \& } T((x_k)_{k \in N}) = \theta\}$ for $\theta \in R$, then all conditions in Definition 2.4 will be satisfied.

In the sequel we will need the well known fact from the probability theory (see, for example, [3], p. 390).

Lemma 2.1. (Kolmogorov's strong law of large numbers) *Let X_1, X_2, \dots be a sequence of independent identically distributed random variables defined on the probability space (Ω, \mathcal{F}, P) . If these random variables have a finite expectation m (i.e., $E(X_1) = E(X_2) = \dots = m < \infty$), then the following condition*

$$P(\{\omega : \lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n X_k(\omega) = m\}) = 1$$

holds true.

3. ESTIMATION OF PARAMETERS OF WIENER PROCESS WITH DRIFT

3.1. Estimation of a parameter of the bombardment force σ in Wiener model with drift. The purpose of the present subsection is to estimate a parameter of the bombardment force σ by water molecules acting on a particle suspended in water under assumption that we know results of observations on placements of the particle at moment t_0 , a parameter of the drift μ and an initial position x_0 .

Theorem 3.1.1 *For $t > 0$, $x_0 \in R$, $\mu \in R$ and $\sigma > 0$, let's $\gamma_{(t, x_0, \mu, \sigma)}$ be a Gaussian probability measure in R with the mean $m_t = x_0 + \mu t$ and the variance $\sigma_t^2 = \sigma^2 t$. Assuming that parameters t , x_0 and μ are fixed, denote by γ_{σ^2} the measure $\gamma_{(t, x_0, \mu, \sigma)}$. Let define the estimate $T_n : R^n \rightarrow R$ by the following formula*

$$T_n((z_k)_{1 \leq k \leq n}) = \frac{\sum_{k=1}^n (z_k - x_0 - t\mu)^2}{nt}. \quad (3.1.1)$$

Then we get

$$\gamma_{\sigma^2}^\infty\{(z_k)_{k \in N} : (z_k)_{k \in N} \in R^\infty \text{ \& } \lim_{n \rightarrow \infty} T_n((z_k)_{1 \leq k \leq n}) = \sigma^2\} = 1, \quad (3.1.2)$$

provided that T_n is a consistent estimator of a parameter of the bombardment force σ in Wiener model with drift for the family of probability measures $(\gamma_{\sigma^2}^\infty)_{\sigma^2 > 0}$.

Proof. Let's consider probability space (Ω, \mathcal{F}, P) , where $\Omega = R^\infty$, $\mathcal{F} = B(R^\infty)$, $P = \gamma_{\sigma^2}^\infty$.

For $k \in N$ we consider k -th projection Pr_k defined on R^∞ by

$$Pr_k((x_i)_{i \in N}) = x_k \quad (3.1.3)$$

for $(x_i)_{i \in N} \in R^\infty$.

It is obvious that $(Pr_k)_{k \in N}$ is sequence of independent Gaussian random variables with $m_t = x_0 + \mu t$ and the variance $\sigma_t^2 = \sigma^2 t$. It is obvious that $(\frac{(Pr_k - x_0 - t\mu)^2}{t})_{k \in N}$ is the sequence of independent equally distributed random variables with mean σ^2 .

By use Kolmogorov Strong Law of Large numbers we get

$$\gamma_{\sigma^2}^\infty\{(z_i)_{i \in N} \in R^\infty \text{ \& } \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{(Pr_k((z_i)_{i \in N}) - x_0 - t\mu)^2}{tn} = \sigma^2\} = 1, \quad (3.1.4)$$

which implies

$$\gamma_{\sigma^2}^\infty\{(z_i)_{i \in N} \in R^\infty \text{ \& } \lim_{n \rightarrow \infty} T_n((z_k)_{1 \leq k \leq n}) = \sigma^2\} = 1.$$

□

Remark 3.1.1 By use Definition 2.1, Remark 2.1 and Theorem 3.1.1 we deduce that T_n is a consistent estimator of a parameter of the bombardment force σ in the sense of convergence in probability for the statistical structure $(\gamma_{\sigma^2})_{\sigma^2 > 0}$ as well T_n is a consistent estimator of a parameter of the bombardment force σ in the sense of convergence in distribution for the statistical structure $(\gamma_{\sigma^2})_{\sigma^2 > 0}$.

Theorem 3.1.2 Suppose that the family of probability measures $(\gamma_{\sigma^2})_{\sigma^2 > 0}$ and the estimators $T_n : R^n \rightarrow R (n \in N)$ come from Theorem 4.1.1. Then the estimators $T^{(0)} : R^\infty \rightarrow R$ and $T^{(1)} : R^\infty \rightarrow R$ defined by

$$T^{(0)}((z_k)_{k \in N}) = \varliminf_{n \rightarrow \infty} T_n((z_k)_{1 \leq k \leq n}) \quad (3.1.5)$$

and

$$T^{(1)}((z_k)_{k \in N}) = \overline{\lim}_{n \rightarrow \infty} T_n((z_k)_{1 \leq k \leq n}). \quad (3.1.6)$$

are infinite-sample consistent estimators of a parameter of the bombardment force σ in Wiener model with drift for the family of probability measures $(\gamma_{\sigma^2}^\infty)_{\sigma^2 > 0}$.

Proof. Note that we have

$$\begin{aligned} & \gamma_{\sigma^2}^\infty\{(z_k)_{k \in N} \in R^\infty \text{ \& } T^{(0)}((z_k)_{k \in N}) = \sigma^2\} \\ &= \gamma_{\sigma^2}^\infty\{(z_k)_{k \in N} \in R^\infty \text{ \& } \varliminf_{n \rightarrow \infty} T_n((z_k)_{1 \leq k \leq n}) = \sigma^2\} \\ &\geq \gamma_{\sigma^2}^\infty\{(z_k)_{k \in N} \in R^\infty \text{ \& } \lim_{n \rightarrow \infty} T_n((z_k)_{1 \leq k \leq n}) = \sigma^2\} = 1, \end{aligned}$$

which means that $T^{(0)}$ is an infinite-sample consistent estimator of a parameter of the bombardment force σ in Wiener model with drift for the family of probability measures $(\gamma_{\sigma^2}^\infty)_{\sigma^2 > 0}$.

Similarly, we have

$$\begin{aligned} & \gamma_{\sigma^2}^\infty \{(z_k)_{k \in N} \in R^\infty \ \& \ T^{(1)}((z_k)_{k \in N}) = \sigma^2\} \\ &= \gamma_{\sigma^2}^\infty \{(z_k)_{k \in N} \in R^\infty \ \& \ \overline{\lim}_{n \rightarrow \infty} T_n((z_k)_{1 \leq k \leq n}) = \sigma^2\} \\ &\geq \gamma_{\sigma^2}^\infty \{(z_k)_{k \in N} \in R^\infty \ \& \ \lim_{n \rightarrow \infty} T_n((z_k)_{1 \leq k \leq n}) = \sigma^2\} = 1, \end{aligned}$$

which means that $T^{(1)}$ is an infinite-sample consistent estimator of a parameter of the bombardment force σ in Wiener model with drift for the family of probability measures $(\gamma_{\sigma^2}^\infty)_{\sigma^2 > 0}$. □

Remark 3.1.2 By use Remark 2.2 we deduce that an existence of infinite sample consistent estimators $T^{(0)}$ and $T^{(1)}$ of a parameter of the bombardment force σ in Wiener model with drift for the family of probability measures $(\gamma_{\sigma^2}^\infty)_{\sigma^2 > 0}$ (cf. Theorem 3.1.2) implies that the family $(\gamma_{\sigma^2}^\infty)_{\sigma^2 > 0}$ is strictly separated.

3.2. Estimation of a parameter of the drift μ in Wiener model with drift.

The purpose of the present subsection is to estimate a parameter of the drift μ in Wiener model with drift under assumption that we know results of observations on placements of the particle at moment t_0 , a parameter of the bombardment force σ by water molecules acting on a particle suspended in water and an initial position x_0 .

Theorem 3.2.1 For $t > 0$, $x_0 \in R$, $\mu \in R$ and $\sigma > 0$, let's $\gamma_{(t, x_0, \mu, \sigma)}$ be a Gaussian probability measure in R with the mean $m_t = x_0 + \mu t$ and the variance $\sigma_t^2 = \sigma^2 t$. Assuming that parameters t , x_0 and σ are fixed, for $\mu \in R$ denote by γ_μ the measure $\gamma_{(t, x_0, \mu, \sigma)}$. Let define the estimate $T_n^* : R^n \rightarrow R$ by the following formula

$$T_n^*((z_k)_{1 \leq k \leq n}) = \frac{\sum_{k=1}^n (z_k - x_0)}{nt}. \quad (3.2.1)$$

Then we get

$$\gamma_\mu^\infty \{(z_k)_{k \in N} : (z_k)_{k \in N} \in R^\infty \ \& \ \lim_{n \rightarrow \infty} T_n^*((z_k)_{1 \leq k \leq n}) = \mu\} = 1, \quad (3.2.2)$$

for $\mu \in R$ provided that T_n is a consistent estimator of a parameter of the drift μ in Wiener model with drift for the family of probability measures $(\gamma_\mu^\infty)_{\mu \in R}$.

Proof. Let's consider probability space (Ω, \mathcal{F}, P) , where $\Omega = R^\infty$, $\mathcal{F} = B(R^\infty)$, $P = \gamma_{\sigma^2}^\infty$.

For $k \in N$ we consider k -th projection Pr_k defined on R^∞ by

$$Pr_k((x_i)_{i \in N}) = x_k \quad (3.2.3)$$

for $(x_i)_{i \in N} \in R^\infty$.

It is obvious that $(Pr_k)_{k \in N}$ is sequence of independent Gaussian random variables with $m_t = x_0 + \mu t$ and the variance $\sigma_t^2 = \sigma^2 t$. It is obvious that $(\frac{Pr_k - x_0}{t})_{k \in N}$ is the sequence of independent equally distributed random variables with mean μ .

By use Kolmogorov Strong Law of Large numbers we get

$$\gamma_\mu^\infty \{(z_i)_{i \in N} \in R^\infty \text{ \& } \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{Pr_k - x_0}{tn} = \mu\} = 1, \quad (3.2.4)$$

which implies

$$\gamma_\mu^\infty \{(z_i)_{i \in N} \in R^\infty \text{ \& } \lim_{n \rightarrow \infty} T_n^*((z_k)_{1 \leq k \leq n}) = \mu\} = 1.$$

□

Remark 3.2.1 By use Definition 2.1, Remark 2.1 and Theorem 3.2.1 we deduce that T_n^* is a consistent estimator of a parameter of the drift μ in Wiener model with drift in the sense of convergence in probability for the statistical structure $(\gamma_\mu)_{\mu \in R}$ as well T_n^* is a consistent estimator of a parameter of the drift μ in Wiener model with drift in the sense of convergence in distribution for the statistical structure $(\gamma_\mu)_{\mu \in R}$.

Theorem 3.2.2 Suppose that the family of probability measures $(\gamma_\mu)_{\mu \in R}$ and the estimators $T_n^* : R^n \rightarrow R (n \in N)$ come from Theorem 4.2.1. Then the estimators $T_*^{(0)} : R^\infty \rightarrow R$ and $T_*^{(1)} : R^\infty \rightarrow R$ defined by

$$T^{(0)}((z_k)_{k \in N}) = \varliminf_{n \rightarrow \infty} T_n^*((z_k)_{1 \leq k \leq n}) \quad (3.2.5)$$

and

$$T_*^{(1)}((z_k)_{k \in N}) = \overline{\lim}_{n \rightarrow \infty} T_n^*((z_k)_{1 \leq k \leq n}). \quad (3.2.6)$$

are infinite-sample consistent estimators of a parameter of the drift μ in Wiener model with drift for the family of probability measures $(\gamma_\mu)_{\mu \in R}$.

Proof. Note that we have

$$\begin{aligned} & \gamma_\mu^\infty \{(z_k)_{k \in N} \in R^\infty \text{ \& } T_*^{(0)}((z_k)_{k \in N}) = \mu\} \\ &= \gamma_\mu^\infty \{(z_k)_{k \in N} \in R^\infty \text{ \& } \varliminf_{n \rightarrow \infty} T_n^*((z_k)_{1 \leq k \leq n}) = \mu\} \\ &\geq \gamma_\mu^\infty \{(z_k)_{k \in N} \in R^\infty \text{ \& } \lim_{n \rightarrow \infty} T_n^*((z_k)_{1 \leq k \leq n}) = \mu\} = 1, \end{aligned}$$

which means that $T_*^{(0)}$ is an infinite-sample consistent estimator of a parameter of the drift μ in Wiener model with drift for the family of probability measures $(\gamma_\mu^\infty)_{\mu \in R}$.

Similarly, we have

$$\begin{aligned} & \gamma_\mu^\infty \{(z_k)_{k \in N} \in R^\infty \text{ \& } T_*^{(1)}((z_k)_{k \in N}) = \mu\} \\ &= \gamma_\mu^\infty \{(z_k)_{k \in N} \in R^\infty \text{ \& } \overline{\lim}_{n \rightarrow \infty} T_n^*((z_k)_{1 \leq k \leq n}) = \mu\} \\ &\geq \gamma_\mu^\infty \{(z_k)_{k \in N} \in R^\infty \text{ \& } \lim_{n \rightarrow \infty} T_n^*((z_k)_{(z_k)_{1 \leq k \leq n}}) = \mu\} = 1, \end{aligned}$$

which means that $T_*^{(1)}$ is an infinite-sample consistent estimator of a parameter of the drift μ in Wiener model with drift for the family of probability measures $(\gamma_\mu^\infty)_{\mu \in R}$.

□

Remark 3.2.2 By use Remark 2.2 we deduce that an existence of infinite sample consistent estimators $T_*^{(0)}$ and $T_*^{(1)}$ of a parameter of the drift μ in Wiener model with drift for the family of probability measures $(\gamma_\mu^\infty)_{\mu \in R}$ (cf. Theorem 3.2.1) implies that the family $(\gamma_\mu^\infty)_{\mu \in R}$ is strictly separated.

3.3. Estimation of an initial position of the particle suspended in water in Wiener model with drift. The purpose of the present subsection is to estimate an initial position of the particle suspended in water under assumption that we know results of observations on placements of the particle at moment t_0 , a parameters of the drift μ and a parameter of the bombardment force σ by water molecules acting on a particle.

Theorem 3.3.1 For $t > 0$, $x_0 \in R$, $\mu \in R$ and $\sigma > 0$, let's $\gamma_{(t, x_0, \mu, \sigma)}$ be a Gaussian probability measure in R with the mean $m_t = x_0 + \mu t$ and the variance $\sigma_t^2 = \sigma^2 t$. Assuming that parameters t , μ and σ are fixed, denote by γ_{x_0} the measure $\gamma_{(t, x_0, \mu, \sigma)}$. Let define the estimate $T_n^{**} : R^n \rightarrow R$ by the following formula

$$T_n^{**}((z_k)_{1 \leq k \leq n}) = \frac{\sum_{k=1}^n (z_k - t\mu)^2}{n}. \quad (3.3.1)$$

Then we get

$$\gamma_{x_0}^\infty\{(z_k)_{k \in N} : (z_k)_{k \in N} \in R^\infty \text{ \& \; } \lim_{n \rightarrow \infty} T_n^{**}((z_k)_{1 \leq k \leq n}) = x_0\} = 1, \quad (3.3.2)$$

for $x_0 \in R$ provided that T_n^{**} is a consistent estimator of an initial position of the particle suspended in water in Wiener model with drift for the family of probability measures $(\gamma_{x_0}^\infty)_{x_0 \in R}$.

Proof. Let's consider probability space (Ω, \mathcal{F}, P) , where $\Omega = R^\infty$, $\mathcal{F} = B(R^\infty)$, $P = \gamma_{x_0}^\infty$.

For $k \in N$ we consider k -th projection Pr_k defined on R^∞ by

$$Pr_k((x_i)_{i \in N}) = x_k \quad (3.3.3)$$

for $(x_i)_{i \in N} \in R^\infty$.

It is obvious that $(Pr_k)_{k \in N}$ is sequence of independent Gaussian random variables with $m_t = x_0 + \mu t$ and the variance $\sigma_t^2 = \sigma^2 t$. It is obvious that $(Pr_k - t\mu)_{k \in N}$ is the sequence of independent equally distributed random variables with mean x_0 .

By use Kolmogorov Strong Law of Large numbers we get

$$\gamma_{x_0}^\infty\{(z_i)_{i \in N} \in R^\infty \text{ \& \; } \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{(Pr_k((z_i)_{i \in N}) - t\mu)^2}{n} = x_0\} = 1, \quad (3.3.4)$$

which implies

$$\gamma_{x_0}^\infty\{(z_i)_{i \in N} \in R^\infty \text{ \& \; } \lim_{n \rightarrow \infty} T_n^{**}((z_k)_{1 \leq k \leq n}) = x_0\} = 1.$$

□

Remark 3.3.1 By use Definition 3.1, Remark 2.1 and Theorem 3.3.1 we deduce that T^{**} is a consistent estimator of an initial position x_0 of the particle suspended in water in Wiener model with drift in the sense of convergence in probability for the statistical structure $(\gamma_{x_0})_{x_0 \in R}$ as well T^{**} is a consistent estimator of an initial position x_0 in the sense of convergence in distribution for the statistical structure $(\gamma_{x_0})_{x_0 \in R}$.

Theorem 3.3.2 *Suppose that the family of probability measures $(\gamma_{x_0})_{x_0 \in R}$ and the estimators $T^{**} : R^n \rightarrow R (n \in N)$ come from Theorem 3.3.1. Then the estimators $T_{***}^{(0)} : R^\infty \rightarrow R$ and $T_{***}^{(1)} : R^\infty \rightarrow R$ defined by*

$$T_{***}^{(0)}((z_k)_{k \in N}) = \varliminf_{n \rightarrow \infty} T^{**}((z_k)_{1 \leq k \leq n}) \quad (3.3.5)$$

and

$$T_{***}^{(1)}((z_k)_{k \in N}) = \overline{\lim}_{n \rightarrow \infty} T^{**}((z_k)_{1 \leq k \leq n}). \quad (3.3.6)$$

are infinite-sample consistent estimators of an initial position x_0 of the particle suspended in water in Wiener model with drift for the family of probability measures $(\gamma_{x_0})_{x_0 \in R}$.

Proof. Note that we have

$$\begin{aligned} & \gamma_{x_0}^\infty \{ (z_k)_{k \in N} \in R^\infty \ \& \ T_{***}^{(0)}((z_k)_{k \in N}) = x_0 \} \\ &= \gamma_{x_0}^\infty \{ (z_k)_{k \in N} \in R^\infty \ \& \ \varliminf_{n \rightarrow \infty} T^{**}((z_k)_{1 \leq k \leq n}) = x_0 \} \\ &\geq \gamma_{x_0}^\infty \{ (z_k)_{k \in N} \in R^\infty \ \& \ \lim_{n \rightarrow \infty} T^{**}((z_k)_{1 \leq k \leq n}) = x_0 \} = 1, \end{aligned}$$

which means that $T_{***}^{(0)}$ is an infinite-sample consistent estimator of an initial position x_0 of the particle suspended in water in Wiener model with drift for the family of probability measures $(\gamma_{x_0})_{x_0 \in R}$.

Similarly, we have

$$\begin{aligned} & \gamma_{x_0}^\infty \{ (z_k)_{k \in N} \in R^\infty \ \& \ T_{***}^{(1)}((z_k)_{k \in N}) = x_0 \} \\ &= \gamma_{x_0}^\infty \{ (z_k)_{k \in N} \in R^\infty \ \& \ \overline{\lim}_{n \rightarrow \infty} T^{**}((z_k)_{1 \leq k \leq n}) = x_0 \} \\ &\geq \gamma_{x_0}^\infty \{ (z_k)_{k \in N} \in R^\infty \ \& \ \lim_{n \rightarrow \infty} T^{**}((z_k)_{1 \leq k \leq n}) = x_0 \} = 1, \end{aligned}$$

which means that $T_{***}^{(1)}$ is an infinite-sample consistent estimator of an initial position x_0 of the particle suspended in water in Wiener model with drift for the family of probability measures $(\gamma_{x_0})_{x_0 \in R}$. □

Remark 3.3.2 By use Remark 2.2 we deduce that an existence of infinite sample consistent estimators $T^{(0)}$ and $T^{(1)}$ of an initial position x_0 of the particle suspended in water in Wiener model with drift for the family of probability measures $(\gamma_{x_0})_{x_0 \in R}$ (cf. Theorem 3.3.2) implies that the family $(\gamma_{x_0})_{x_0 \in R}$ is strictly separated.

4. SIMULATIONS, CALCULATIONS AND ANIMATIONS OF THE WIENER PROCESS WITH DRIFT

In this section we present some programmes in Matlab for simulation and animation of the Wiener process with drift. In preparation of these programmes we have used main approaches and technique introduced in [5].

The simulation of the Wiener process with drift can be obtained as follows:

$$x_t = x_0 + \mu t + \sigma(d_0 t + \sqrt{2} \sum_{n=1}^{\infty} d_n \frac{\sin \pi n t}{\pi n}), \quad (4.4.1)$$

where $(d_k)_{k \in N}$ is realization of independent standard Gaussian random variables.

If $(d_n^{(k)})_{n \in \mathcal{N}}$ ($k \in \mathcal{N}$) is k -th realizations of the infinite family of independent standard Gaussian random variables, then the value of the k -th trajectory of the Wiener process with drift at moment t will be

$$z_k = x_0 + \mu t + \sigma(d_0^{(k)}t + \sqrt{2} \sum_{n=1}^{\infty} d_n^{(k)} \frac{\sin \pi n t}{\pi n}) \quad (4.4.2)$$

for each $k \in N$.

In our simulation we use MatLab command **random('Normal',0,1,p, q)** which generates p coppies $(d_n^{(k)})_{1 \leq n \leq q}$ ($1 \leq k \leq p$) of realizations of the finite family of independent standard Gaussian random variables of lenght q .

In our simulation we consider the following approximation

$$z_k = x_0 + \mu t + \sigma(d_0 t + \sqrt{2} \sum_{n=1}^{1000} d_n \frac{\sin \pi n t}{\pi n}) \quad (4.4.3)$$

for $1 \leq k \leq 100$.

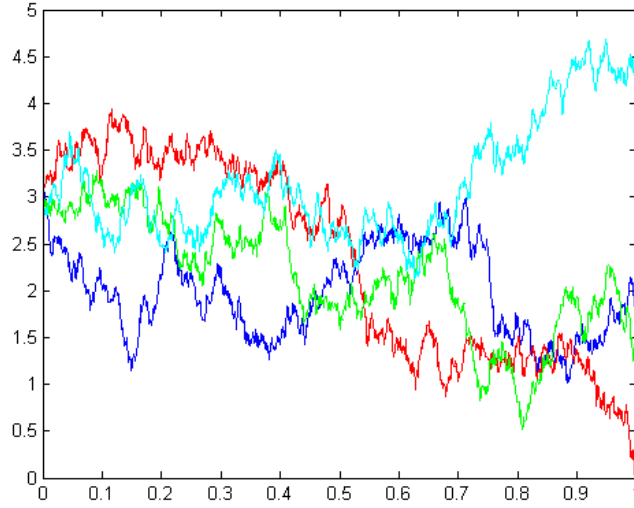


FIGURE 1. Four trajectories of the Wiener process with drift when $x_0 = 3, \mu = -1, \sigma = 2$.

The programm in MatLab giving a simulation of four trajectories of the Wiener process with drift with parameters $\mu = -1, \sigma = 3, x_0 = 3$ has the following form:

```
x1 = random('Normal', 0, 1, 4, 1001);
m = -1;
s = 2;
x0 = 3;
x = 0 : 0.0001 : 1;
Y10 = x0 + m * x;
Y11 = x1(1,1) * x;
```

```

Y20 = x0 + m * x;
Y21 = x1(2, 1) * x;
Y30 = x0 + m * x;
Y31 = x1(3, 1) * x;
Y40 = x0 + m * x;
Y41 = x1(4, 1) * x;
for k = 1 : 1000
Y11 = Y11 + sqrt(2) * x1(1, k + 1) * sin(pi * k * x)/(pi * k);
Y21 = Y21 + sqrt(2) * x1(2, k + 1) * sin(pi * k * x)/(pi * k);
Y31 = Y31 + sqrt(2) * x1(3, k + 1) * sin(pi * k * x)/(pi * k);
Y41 = Y41 + sqrt(2) * x1(4, k + 1) * sin(pi * k * x)/(pi * k);
end
X1 = x;
Y1 = Y10 + s * Y11;
X2 = x;
Y2 = Y20 + s * Y21;
X3 = x;
Y3 = Y30 + s * Y31;
X4 = x;
Y4 = Y40 + s * Y41;
plot(X1, Y1, 'b', X2, Y2, 'r-', X3, Y3, 'g', X4, Y4, 'c', 'LineWidth', 1)

```

Below we present some numerical results obtaining by using MatLab and Microsoft Excel. In our simulation

- (i) n denotes the number of trials;
- (ii) $x_0 = 3$ is an initial position of the particle suspended in water;
- (iii) $\mu = -1$ is the equilibrium or mean position supported by fundamentals;
- (iv) $\sigma = 2$ is the parameter of bombardment in Wiener process with drift ;
- (v) $t = 0.5$ is the moment of the observation on the Wiener process with drift ;
- (vi) z_k is the value of the k -th trajectory of Wiener process with drift at moment $t = 0.5$ (see, Figure 1 and Table 4.1).

Table 4.1. The value z_k of the the Wiener process with drift at moment $t = 0.5$ when $x_0 = 3, \mu = -1, \sigma = 2$.

k	z_k	k	z_k	k	z_k	k	z_k	k	z_k
1	4.0991	21	2.7068	41	4.3571	61	3.5272	81	7.0549
2	1.6842	22	2.243	42	3.2793	62	2.8292	82	2.5627
3	2.9422	23	3.5946	43	3.2422	63	1.8519	83	1.5592
4	4.5744	24	4.6402	44	2.7395	64	-0.5223	84	3.8465
5	2.0157	25	2.2703	45	0.682	65	0.7602	85	0.8471
6	2.8821	26	2.681	46	3.4298	66	4.3529	86	2.8475
7	4.6284	27	2.9075	47	2.6592	67	1.1265	87	0.5144
8	1.654	28	3.3659	48	3.2093	68	1.3525	88	2.3558
9	0.9561	29	4.5149	49	0.1074	69	3.7889	89	2.4793
10	-0.5407	30	-1.3528	50	1.5462	70	3.2284	90	2.0639
11	1.98941	31	3.4294	51	1.0896	71	1.1392	91	1.166
12	4.3462	32	4.0825	52	2.1108	72	2.8833	92	3.2738
13	3.455	33	2.3837	53	2.9175	73	2.3093	93	0.4878
14	3.2235	34	3.3037	54	2.9352	74	1.4444	94	2.9504
15	2.1299	35	4.7552	55	3.5531	75	-0.23	95	2.4767
16	1.7554	36	2.1509	56	2.2376	76	1.1718	96	3.6022
17	3.9263	37	4.3749	57	1.6557	77	1.3588	97	1.6005
18	2.4328	38	2.6403	58	2.0076	78	1.5024	98	3.5125
19	2.9972	39	1.4087	59	0.8997	79	2.2622	99	2.2026
20	2.59	40	4.178	60	0.1873	80	3.3423	100	3.3646

Table 4.2. The value of the statistic T_n for the sample $(z_k)_{1 \leq k \leq n}$ ($n = 5i : 1 \leq i \leq 20$) from the Table 4.1.

n	T_n	σ^2	n	T_n	σ^2
5	3.182349256	4	55	3.463530492	4
10	4.995431962	4	60	3.472692797	4
15	4.029170383	4	65	3.628477887	4
20	3.30673624	4	70	3.621538801	4
25	3.120603594	4	75	3.662829637	4
30	3.924002323	4	80	3.554594699	4
35	3.884200678	4	85	3.96153126	4
40	3.781846824	4	90	3.836441056	4
45	3.715840465	4	95	3.774116249	4
50	3.665346353	4	100	3.663112062	4

Remark 4.1 By use results of calculations placed in the Table 4.2, we see that the consistent estimator T_n works successfully.

Table 4.3. The value of the statistic T_n^* for the sample $(z_k)_{1 \leq k \leq n}$ ($n = 5i : 1 \leq i \leq 20$) from the Table 4.1.

n	T_n^*	μ	n	T_n^*	μ
5	0.12624	-1	55	-0.47716	-1
10	-1.0209	-1	60	-0.704466667	-1
15	-0.6614	-1	65	-0.851932308	-1
20	-0.62588	-1	70	-0.82396	-1
25	-0.464312	-1	75	-0.967797333	-1
30	-0.57916	-1	80	-1.0413725	-1
35	-0.327594286	-1	85	-0.959635294	-1
40	-0.299005	-1	90	-1.011635556	-1
45	-0.296888889	-1	95	-1.056187368	-1
50	-0.429124	-1	100	-1.01773	-1

Remark 4.2 By use results of calculations placed in the Table 4.2, we see that the consistent estimator T_n^* has a tendetion will come nearer to $\mu = -1$ as soon as the number of trials increases.

The following program gives animation of the Wiener process with drift over the time interval $[0, 1]$ when $x_0 = 3, \mu = -1, \sigma = 2$.

```

N = 100;
x1 = random('Normal', 0, 1, N, 10001);
m = -1;
s = 2;
x0 = 3;
x = 0 : 0.0001 : 1;
Y0 = x0 + m * x;
Y1 = x1(1) * x;
for s = 1 : N
for k = 1 : 1000
Y1 = Y1 + sqrt(2) * x1(s, k + 1) * sin(pi * k * x) / (pi * k);
end
X = x;
Y = Y0 + s * Y1;
plot (X, Y, 'r', 'LineWidth', 1)
drawnow;
pause(1);
end

```

An animation given by this programm applies N different trajectories which are defined by N coppies of realizations of the finite family of independent standard Gaussian random variables of lenght 1001 generated by Matlab operator $x1 = random('Normal', 0, 1, N, 10001)$;

5. FURTHER INVESTIGATIONS

Suppose that $(z_k^{(1)})_{k \in N}$ and $(z_k^{(2)})_{k \in N}$ are results of observations on the $2k$ -th and $2k + 1$ -th trajectories of the Wiener process with drift at moments t_1 and t_2 , respectively. Note that having such an information we can estimate unknown parameters x_0, μ, σ for Wiener process with drift .

The first step in this direction is made by the following proposition.

Theorem 5.1 For $t > 0$, $x_0 \in R$, $\mu \in R$ and $\sigma > 0$, let's $\gamma_{(t,x_0,\mu,\sigma)}$ be a Gaussian probability measure in R with the mean $m_t = x_0 + \mu t$ and the variance $\sigma_t^2 = \sigma^2 t$. Assuming that the parameter σ is fixed, for $t_1 > 0$, $t_2 > 0$, $x_0 \in R$, $\mu \in R$ denote by $\gamma_{(x_0,\mu,t_1)}$ and $\gamma_{(x_0,\mu,t_2)}$ the measure $\gamma_{(t_1,x_0,\mu,\sigma)}$ and $\gamma_{(t_2,x_0,\mu,\sigma)}$, respectively.

We put $\gamma_{(x_0,\mu)} = \gamma_{(x_0,\mu,t_1)} \times \gamma_{(x_0,\mu,t_2)}$. Let define the estimate $\mathbf{T}_n : (R^2)^n \rightarrow R^2$ by the following formula

$$\mathbf{T}_n((z_k^{(1)}, z_k^{(2)})_{1 \leq k \leq n}) = \left(\frac{\sum_{k=1}^n (t_2 z_k^{(1)} - t_1 z_k^{(2)})}{n(t_2 - t_1)}, \frac{\sum_{k=1}^n (z_k^{(2)} - z_k^{(1)})}{n(t_2 - t_1)} \right). \quad (5.1)$$

Then we get

$$\begin{aligned} & \gamma_{(x_0,\mu)}^\infty \{ (z_k^{(1)}, z_k^{(2)})_{k \in N} : (z_k^{(1)}, z_k^{(2)})_{k \in N} \in (R^2)^\infty \\ & \& \lim_{n \rightarrow \infty} \mathbf{T}_n(((z_k^{(1)}, z_k^{(2)})_{1 \leq k \leq n}) = (x_0, \mu) \} = 1, \end{aligned} \quad (5.2)$$

for $(x_0, \mu) \in R^2$ provided that \mathbf{T}_n is a consistent estimator of a parameter (x_0, μ) for the family of probability measures $(\gamma_{(x_0,\mu)}^\infty)_{(x_0,\mu) \in R^2}$.

Proof. Let's consider probability space (Ω, \mathcal{F}, P) , where $\Omega = (R^2)^\infty$, $\mathcal{F} = B((R^2)^\infty)$, $P = \gamma_{(x_0,\mu)}^\infty$.

For $k \in N$ we consider k -th projection $\mathbf{Pr}_k = (Pr_k^{(1)}, Pr_k^{(2)})$ defined on $(R^2)^\infty$ by

$$\mathbf{Pr}_k((x_i^{(1)}, x_i^{(2)})_{i \in N}) = (x_k^{(1)}, x_k^{(2)}) \quad (5.3)$$

for $(x_i^{(1)}, x_i^{(2)})_{i \in N} \in (R^2)^\infty$.

It is obvious that $Pr_k^{(1)}((x_i^{(1)}, x_i^{(2)})_{i \in N}) = x_k^{(1)}$ and $Pr_k^{(2)}((x_i^{(1)}, x_i^{(2)})_{i \in N}) = x_k^{(2)}$ are also projection operators for $(x_i^{(1)}, x_i^{(2)})_{i \in N} \in (R^2)^\infty$.

It is obvious that $(Pr_k^{(1)})_{k \in N}$ is sequence of independent one-dimensional Gaussian random variables with expectaion equal to $x_0 + \mu t_1$ and the variance equal to $\sigma^2 t_1$. Similarly, $(Pr_k^{(2)})_{k \in N}$ is sequence of independent one-dimensional Gaussian random variables with expectaion equal to $x_0 + \mu t_2$ and the variance equal to $\sigma^2 t_2$.

Note that by use Kolmogorov strong law of large numbers, we have

$$\gamma_{(x_0,\mu,t_1)}^\infty (\{(z_k^{(1)})_{k \in N} : (z_k^{(1)})_{k \in N} \in R^\infty \& \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n z_k^{(1)}}{n} = x_0 + \mu t_1\}) = 1 \quad (5.4)$$

and

$$\gamma_{(x_0,\mu,t_2)}^\infty (\{(z_k^{(2)})_{k \in N} : (z_k^{(2)})_{k \in N} \in R^\infty \& \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n z_k^{(2)}}{n} = \mu + \mu t_2\}) = 1 \quad (5.5)$$

We have

$$\begin{aligned} & \gamma_{(x_0,\mu)}^\infty \{ (z_k^{(1)}, z_k^{(2)})_{k \in N} : (z_k^{(1)}, z_k^{(2)})_{k \in N} \in (R^2)^\infty \\ & \& \lim_{n \rightarrow \infty} \mathbf{T}_n(((z_k^{(1)}, z_k^{(2)})_{1 \leq k \leq n}) = (x_0, \mu) \} \\ & = \gamma_{(x_0,\mu)}^\infty \{ (z_k^{(1)}, z_k^{(2)})_{k \in N} : (z_k^{(1)}, z_k^{(2)})_{k \in N} \in (R^2)^\infty \\ & \& \lim_{n \rightarrow \infty} T_n^{(1)}(((z_k^{(1)}, z_k^{(2)})_{1 \leq k \leq n} = x_0 \& \lim_{n \rightarrow \infty} T_n^{(2)}(((z_k^{(1)}, z_k^{(2)})_{1 \leq k \leq n} = \mu) \} \\ & = \gamma_{(x_0,\mu)}^\infty \{ (z_k^{(1)}, z_k^{(2)})_{k \in N} : (z_k^{(1)}, z_k^{(2)})_{k \in N} \in (R^2)^\infty \end{aligned} \quad (5.6)$$

$$\begin{aligned}
& \& \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n (t_2 z_k^{(1)} - t_1 z_k^{(2)})}{n(t_2 - t_1)} = x_0 \& \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n (z_k^{(2)} - z_k^{(1)})}{n(t_2 - t_1)} = \mu \} \\
& = \gamma_{(x_0, \mu)}^\infty \{ (z_k^{(1)}, (z_k^{(2)})_{k \in N} : (z_k^{(1)}, (z_k^{(2)})_{k \in N} \in (R^2)^\infty \\
& \& \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n z_k^{(1)}}{n} = x_0 \& \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n z_k^{(2)}}{n} = \mu \} \\
& = \gamma_{(x_0, \mu, t_1)}^\infty (\{ (z_k^{(1)})_{k \in N} : (z_k^{(1)})_{k \in N} \in (R)^\infty \& \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n z_k^{(1)}}{n} = x_0 + \mu t_1 \}) \\
& \times \gamma_{(x_0, \mu, t_2)}^\infty (\{ (z_k^{(2)})_{k \in N} : (z_k^{(2)})_{k \in N} \in (R)^\infty \& \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n z_k^{(2)}}{n} = \mu + \mu t_2 \}) \\
& \text{(By use (5.4)-(5.5), we have)} \\
& = 1 \times 1 = 1.
\end{aligned}$$

□

Theorem 5.2 Suppose that the family of probability measures $(\gamma_{(x_0, \mu)}^\infty)_{(x_0, \mu) \in R^2}$ and the estimators $\mathbf{T}_n : (R^2)^n \rightarrow R^2$ come from Theorem 5.1. Then the estimators $\mathbf{T}^{(0)} : R^\infty \rightarrow R$ and $\mathbf{T}^{(1)} : R^\infty \rightarrow R$ defined by

$$\mathbf{T}^{(0)}((z_k^{(1)}, z_k^{(2)})_{k \in N}) = \varliminf_{n \rightarrow \infty} \mathbf{T}_n((z_k^{(1)}, z_k^{(2)})_{1 \leq k \leq n}) \quad (5.7)$$

and

$$\mathbf{T}^{(1)}((z_k^{(1)}, z_k^{(2)})_{k \in N}) = \overline{\lim}_{n \rightarrow \infty} \mathbf{T}_n((z_k^{(1)}, z_k^{(2)})_{1 \leq k \leq n}). \quad (5.8)$$

are infinite-sample consistent estimators of a parameter (x_0, μ) for the family of probability measures $(\gamma_{(x_0, \mu)}^\infty)_{(x_0, \mu) \in R^2}$.

Proof. By using (5.2), we get

$$\begin{aligned}
& \gamma_{(x_0, \mu)}^\infty \{ (z_k^{(1)}, z_k^{(2)})_{k \in N} \in (R^2)^\infty \& \mathbf{T}^{(0)}((z_k^{(1)}, z_k^{(2)})_{k \in N}) = (x_0, \mu) \} \\
& = \gamma_{(x_0, \mu)}^\infty \{ (z_k^{(1)}, z_k^{(2)})_{k \in N} \in (R^2)^\infty \& \varliminf_{n \rightarrow \infty} \mathbf{T}_n((z_k^{(1)}, z_k^{(2)})_{1 \leq k \leq n}) = (x_0, \mu) \} \\
& \geq \gamma_{(x_0, \mu)}^\infty \{ (z_k^{(1)}, z_k^{(2)})_{k \in N} \in (R^2)^\infty \& \lim_{n \rightarrow \infty} \mathbf{T}_n((z_k^{(1)}, z_k^{(2)})_{1 \leq k \leq n}) = (x_0, \mu) \} = 1,
\end{aligned}$$

which means that $\mathbf{T}^{(0)}$ is an infinite-sample consistent estimators of a parameter (x_0, μ) for the family of probability measures $(\gamma_{(x_0, \mu)}^\infty)_{(x_0, \mu) \in R^2}$.

Similarly, by using (5.2), we have

$$\begin{aligned}
& \gamma_{(x_0, \mu)}^\infty \{ (z_k^{(1)}, z_k^{(2)})_{k \in N} \in (R^2)^\infty \& \mathbf{T}^{(1)}((z_k^{(1)}, z_k^{(2)})_{k \in N}) = (x_0, \mu) \} \\
& = \gamma_{(x_0, \mu)}^\infty \{ (z_k^{(1)}, z_k^{(2)})_{k \in N} \in (R^2)^\infty \& \overline{\lim}_{n \rightarrow \infty} \mathbf{T}_n((z_k^{(1)}, z_k^{(2)})_{1 \leq k \leq n}) = (x_0, \mu) \} \\
& \geq \gamma_{(x_0, \mu)}^\infty \{ (z_k^{(1)}, z_k^{(2)})_{k \in N} \in (R^2)^\infty \& \lim_{n \rightarrow \infty} \mathbf{T}_n((z_k^{(1)}, z_k^{(2)})_{1 \leq k \leq n}) = (x_0, \mu) \} = 1,
\end{aligned}$$

which means that $\mathbf{T}^{(1)}$ is also an infinite-sample consistent estimators of a parameter (x_0, μ) for the family of probability measures $(\gamma_{(x_0, \mu)}^\infty)_{(x_0, \mu) \in R^2}$.

□

Remark 5.1 Following Theorems 5.1-5.2, by using the values $(z_k^{(1)})_{k \in N}$ and $(z_k^{(2)})_{k \in N}$ being the results of observations on the $2k$ -th and $2k+1$ -th trajectories of the Wiener process with drift at moments t_1 and t_2 , respectively, we can estimate parameters x_0 and μ . So an estimation of the parameter σ is reduced to the case described in Theorem 3.1.

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